

The Effect of Nuclear Rotation on the Collective Transport Coefficients

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April 12, 2001

Abstract

We have examined the influence of rotation on the potential energy and the transport coefficients of the collective motion (friction and mass coefficients). For axially symmetric deformation of nucleus ^{224}Th we found that at excitations corresponding to temperatures $T \geq 1 \text{ MeV}$ the shell correction to the liquid drop energy practically does not depend on the angular rotation. The friction and mass coefficients obtained within the linear response theory for the same nucleus at temperatures larger than 2 MeV are rather stable with respect to rotation *provided* that the contributions from spurious states arising due to the violation of rotation symmetry are removed. At smaller excitations both friction and mass parameters corresponding to the elongation mode are growing functions of rotational frequency ω_{rot} .

Keywords: collective motion, rotating nuclei, linear response theory, transport coefficients

PACS : 21.10.Pc, 21.60.Cs, 21.60.Ev, 24.10.Pa

1 Introduction

The recent success of Flerov Laboratory, JINR, Dubna in the synthesis of the superheavy compound systems with $Z = 114, 116$ has provoked a considerable theoretical interest to the fusion-fission reactions at low excitation energies. Commonly such reactions are described by solving the Langevin equation [1]-[3] for the collective variables which specify the shape of the nuclear compound system formed in the result of fusion of heavy ions with nuclei. Usually such systems are formed with rather high angular momentum. The effect of rotation on the fusion or fission probability is included at most in the calculation of the macroscopic part of the deformation energy. The possible dependence on rotation of the shell correction as well as friction and inertia is completely ignored. However one might expect the strong dependence on rotation of the transport coefficients since the rotation changes considerably the single-particle spectrum. To the best of our knowledge this effect was analyzed only in [4] where rather strong dependence of friction coefficient on angular velocity was found.

In the present paper we have continued the investigations along the line presented in [4]. Besides the friction coefficient we have paid also attention to the rotational dependence of the

mass parameter and the shell correction. The computations are carried out with two-center shell model [5, 6] which allows for rather flexible parametrization of the shape around the touching point and which was used earlier in dynamical computations [7]. As the compound nucleus we chose the system $^{208}\text{Pb} + ^{16}\text{O} \Rightarrow ^{224}\text{Th}$ for which the experimental information on the temperature dependence of the damping parameter is available [8] and which was studied in [9, 10] without account of the rotation.

Due to technical reasons we had to limit ourselves to the excitations above $T = 1\text{MeV}$. The point is that the rotation violates not only the axial symmetry but the time reversal symmetry too and the *BCS* approximation to pairing interaction breaks down. To account for the pairing accurately one has to solve a kind of Hartree-Fock-Bogolyubov equation what requires rather time consuming computation. For this reason we considered here the excitations corresponding to temperature larger than $T = 1\text{MeV}$ when the pairing can be neglected.

The paper is organized as follows: In Section 2 we quote the main relations of the linear response theory adopted for the rotating nuclei. The quasi-static properties (moment of inertia, liquid drop and shell component of potential energy) are examined in Section 3. The influence of rotation on the response functions and transport coefficients is investigated in Secs.4,5. The special attention here is paid to the elimination of the contribution from spurious states caused by rotation. The main conclusions and open questions are formulated in Summary. The expressions for the matrix elements of \hat{J}_x on two-center oscillator basis wave functions are given in the Appendix.

2 General formalism

By describing the rotating nuclei one usually transforms the Hamiltonian from the laboratory co-ordinate system to the body fixed (or intrinsic) co-ordinate system. In the result, instead of the Hamiltonian $\hat{H}(Q_\mu)$ one has to consider the Routhian operator

$$\hat{R}(Q_\mu, \omega_{rot}) = \hat{H}(Q_\mu) - \omega_{rot} \hat{J}_x \quad (1)$$

with ω_{rot} being the rotational frequency and \hat{J}_x - the projection of angular momentum on the rotation axes (x -axes). The variables Q_μ in (1) are the deformation parameters which specify the shape of the (deformed) mean field.

The energy of rotating nucleus $E = \langle \omega_{rot} | \hat{H} | \omega_{rot} \rangle$ and angular momentum $I = \langle \omega_{rot} | \hat{J}_x | \omega_{rot} \rangle$ are growing function of the rotational frequency ω_{rot} . At small values of ω_{rot} one can use the perturbation theory to obtain

$$E = E_0 + \frac{1}{2} \mathcal{J}_{cran} \omega_{rot}^2 \quad (2)$$

where \mathcal{J}_{cran} is the cranking model moment of inertia. In the approximation of independent particles it is given by

$$\mathcal{J}_{cran} = \hbar^2 \sum_{kj} \frac{n_k - n_j}{\epsilon_j - \epsilon_k} |\langle k | \hat{J}_x | j \rangle|^2 \quad (3)$$

The n_k in (3) are the (temperature dependent) occupation numbers and summation is carried out over the single-particle states k and j . If ω_{rot} is not small the single-particle spectrum ϵ_k and eigen-functions are to be found numerically by solving the eigen-values problem,

$$\hat{R}(Q_\mu, \omega_{rot}) |k\rangle = \epsilon_k(Q_\mu, \omega_{rot}) |k\rangle \quad (4)$$

Like in the case without rotation the transport coefficient of collective motion, the tensors of stiffness $C_{\mu\nu}$, friction $\gamma_{\mu\nu}$ and mass $M_{\mu\nu}$ can be derived within the linear response theory [11, 12] from the so called collective response function $\chi_{\text{coll}}(\omega)$ approximating $\chi_{\text{coll}}(\omega)$ by the response function of damped oscillator

$$[\chi_{\text{coll}}(\omega)]_{\mu\nu} \longrightarrow [\mathbf{k}(-\mathbf{M}\omega^2 - i\boldsymbol{\gamma}\omega + \mathbf{C})^{-1}\mathbf{k}]_{\mu\nu} \quad (5)$$

Here \mathbf{k} is the coupling tensor, see (11) below. The collective response function $\chi_{\text{coll}}(\omega)$ is related to the Fourier transform $\chi_{\mu\nu}(\omega)$ of the intrinsic (causal) response function

$$\tilde{\chi}_{\mu\nu}(t) = \Theta(t) \frac{i}{\hbar} \text{tr} \left(\hat{\rho}_{\text{qs}}(Q_\mu, T) [\hat{F}_\mu^I(t), \hat{F}_\nu^I(0)] \right) \quad (6)$$

by

$$\chi_{\text{coll}}(\omega) = \boldsymbol{\kappa}(\boldsymbol{\kappa} + \boldsymbol{\chi}(\omega))^{-1}\boldsymbol{\chi}(\omega) \quad (7)$$

The $\boldsymbol{\kappa}$ in (7) is the inverse of coupling tensor (11) and operators $\hat{F}_\mu^I(t)$ in (6) are the interaction representation for the derivatives of the Routhian $\hat{R}(Q_\mu, \omega_{\text{rot}})$ with respect to deformation (or rotational frequency ω_{rot}),

$$\hat{F}_\mu^I(t) = e^{-\frac{i}{\hbar}\hat{R}t} \hat{F}_\mu e^{\frac{i}{\hbar}\hat{R}t}, \quad \hat{F}_\mu = \frac{\partial \hat{R}(Q_\mu, \omega_{\text{rot}})}{\partial Q_\mu}, \quad \hat{F}_{\omega_{\text{rot}}} = \frac{\partial \hat{R}(Q_\mu, \omega_{\text{rot}})}{\partial \omega_{\text{rot}}} \quad (8)$$

The average in (6) is calculated with the quasi-static density operator for which the canonical distribution is assumed, $\rho_{\text{qs}}(Q_\mu, \omega_{\text{rot}}, T) \propto \exp(-\hat{R}(Q_\mu, \omega_{\text{rot}})/T)$.

The response function (6) can be used to calculate the deviation of the average value of \hat{F}_μ from its quasi-static value (calculated at some deformation point Q^0), (see [11])

$$\delta\langle \hat{F}_\mu \rangle_t = - \sum_\nu \int_{-\infty}^{\infty} \tilde{\chi}_{\mu\nu}(t-s)(Q_\nu(s) - Q_\nu^0)ds \quad (9)$$

The Fourier transform of (9) reads

$$\delta\langle \hat{F}_\mu \rangle_\omega = - \sum_\nu \chi_{\mu\nu}(\omega) \delta Q_\nu(\omega) \quad (10)$$

with $\delta Q_\nu(\omega)$ being the Fourier transform of $(Q_\nu(s) - Q_\nu^0)$.

The coupling tensor \mathbf{k} (c.f.[11]) is

$$-(\mathbf{k}^{-1})_{\mu\nu} \equiv -\kappa_{\mu\nu} = C_{\mu\nu}(0) + \chi_{\mu\nu}(0) \quad (11)$$

where stiffness $C_{\mu\nu}(0)$ of the free energy $\mathcal{F}(Q, T)$ and static response $\chi_{\mu\nu}(0)$ are defined by the static properties of the system,

$$C_{\mu\nu}(0) \equiv \frac{\partial^2 \mathcal{F}(Q, T)}{\partial Q_\mu \partial Q_\nu} \quad (12)$$

and $\chi_{\mu\nu}(0)$ is the Fourier transform of the intrinsic response function (6) taken at $\omega = 0$.

3 Quasi-static properties

The collective potential energy $E(Q_\mu, I)$ is one of the most essential ingredients appearing in the theory of large scale collective motion. The derivatives of $E(Q_\mu, I)$ with respect to deformation define the collective conservative forces. The second derivatives of $E(Q_\mu, I)$ (stiffness) is used to find the inverse of the coupling tensor $\kappa_{\mu\nu}$ (11) which appears in the collective response function (7).

Like in the case without rotation we will use for calculation of the potential energy the Strutinsky shell correction method [13, 14]. The idea of applying the Strutinsky renormalization to the rotational problem was advanced by Pashkevich et al [15, 16], see also [17]. Following [16] one can express the intrinsic energy $E(Q_\mu, I)$ as

$$E(Q_\mu, I) = E_{LDM}(Q_\mu, I) + \delta R(Q_\mu, I) \quad (13)$$

where $E_{LDM}(Q_\mu, I)$ is the liquid drop energy of rotating nucleus and $\delta R(Q_\mu, I)$ is the shell correction,

$$\delta R = \sum_k \epsilon_k n \left(\frac{\epsilon_k - \lambda}{T} \right) - \int_{-\infty}^{\infty} n \left(\frac{e - \tilde{\lambda}}{T} \right) e \tilde{g}(e) de \quad (14)$$

In the case of finite temperature instead of the shell correction to the intrinsic energy one has to consider the shell correction to the free energy $\delta R \Rightarrow \delta \mathcal{F} = \delta R - T \delta S$, where δS is the shell correction to the entropy, see [19]. The energies ϵ_k in (14) are to be found by diagonalization of the shape dependent Routhian (1) and $\tilde{g}(e)$ is the average density of single-particle states, see [13, 14].

As argued in [16], $E_{LDM}(Q_\mu, I)$ can be represented rather accurately by

$$E_{LDM}(Q_\mu, I) = E_{LDM}(Q_\mu) + I^2/2 \mathcal{J}_{rig}(Q_\mu) \quad (15)$$

Here, $\mathcal{J}_{rig}(Q_\mu)$ is the rigid body moment of inertia for the rotation around x -axes and $E_{LDM}(Q_\mu)$ is the liquid drop energy of non-rotating nucleus.

In the computations presented below we will use the two-center shell model [5, 6] and consider only axially symmetric shapes. Such shapes are specified mainly by two parameters: the distance z_0 between the centers of left and right oscillator potentials and the parameter δ which fix the spheroidal deformation of the "fragments". Below we will consider the case when deformations of left and right fragments are the same, $\delta_1 = \delta_2 = \delta$. Furthermore we will consider here only a one-dimensional path in the deformation space and define $\delta = \delta(z_0)$ looking for the minimum of the total energy at fixed z_0 , see [9].

The comparison of the rigid body and cranking model moment of inertia is shown in Fig.1. It is seen that both methods give rather close results (the pairing was neglected). The dependence of \mathcal{J}_{cran} on the rotation is also not very strong. So, the main source of rotational dependence of $E_{LDM}(Q_\mu, I)$ is the I^2 - term in (15).

The left-hand-side part of Fig.2 shows the rotational dependence of the liquid drop part of deformation energy (we suppose that spherical shape corresponds to $Q_\mu = 0$)

$$E_{LDM}^{def}(Q_\mu, I) = E_{LDM}(Q_\mu, I) - E_{LDM}(Q_\mu = 0, I) \quad (16)$$

As it is seen the rotational dependence of the deformation energy is rather strong. The fission barrier becomes lower due to rotation and disappears completely at $I \approx 60\hbar$ for the nucleus ^{224}Th shown in the figure.

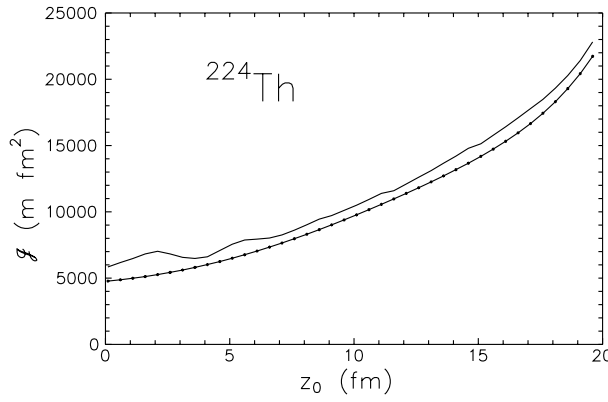


Figure 1: The deformation dependence of the rigid-body \mathcal{J}_{rig} . (curve with dots) and the cranking model \mathcal{J}_{cran} moments of inertia. The deformation parameter z_0 is here the distance between the centers of left and right oscillator potentials of two-center shell model and m is the nucleon mass. The \mathcal{J}_{cran} is computed for the temperature $T = 1 \text{ MeV}$.

The effect of rotation on the fission barriers is known for decades and taken into account nowadays in all computations of the deformation energy. The rotational dependence of the shell correction is less clear. The diagonalization of Routhian (1) is much more time consuming due to the break of axial symmetry as compared with the diagonalization of the non-rotating shell model Hamiltonian. Hence it is assumed usually that this dependence is weak and the shell correction is computed at $\omega_{rot} = 0$ only. To clarify this point we have computed the shell correction for several values of I as a function of deformation along the liquid drop fission valley of ^{224}Th . Indeed, see right-hand-side of Fig.2, the fluctuation of $\delta\mathcal{F}$ is less than 1 MeV for variation of I from zero to $I = 60\hbar$. Very likely such weak dependence of $\delta\mathcal{F}$ on I can be neglected.

The weak dependence of the shell correction on rotation is not so surprising. It was pointed out by Strutinsky [18] that the shell effects are not sensitive to rotation as far as the perturbation $\hbar\omega_{rot}$ is small compared with the spacing $\hbar\Omega_0$ between the gross shells, $\hbar\omega_{rot} \ll \hbar\Omega_0$, with $\hbar\Omega_0 \approx 8 - 10 \text{ MeV}$. For the maximal value of spin $I = 60\hbar$ shown in Fig.2 the $\hbar\omega_{rot} = \hbar I / \mathcal{J} \approx 0.4 \text{ MeV}$ at the saddle and $\hbar\omega_{rot} \approx 0.6 \text{ MeV}$ at the minimum. Both values of $\hbar\omega_{rot}$ are much smaller than $\hbar\Omega_0$. Since the moment of inertia \mathcal{J} increases with growing deformation the $\hbar\omega_{rot}$ gets smaller (for fixed I). This explains, at least partly, why at large deformation the shell correction is less sensitive to rotation as compared with small deformation, see Fig.2.

4 Response functions

The intrinsic response function $\chi_{\mu\nu}(\omega)$ is one of the most simple and important quantity of the linear response theory. The friction and mass coefficients in the so called zero frequency approximation are expressed in terms of derivatives of the intrinsic response functions, see [11]. The intrinsic response function is also an important ingredient of the collective response function (7). So we will look first at the effect of rotation on the intrinsic response function.

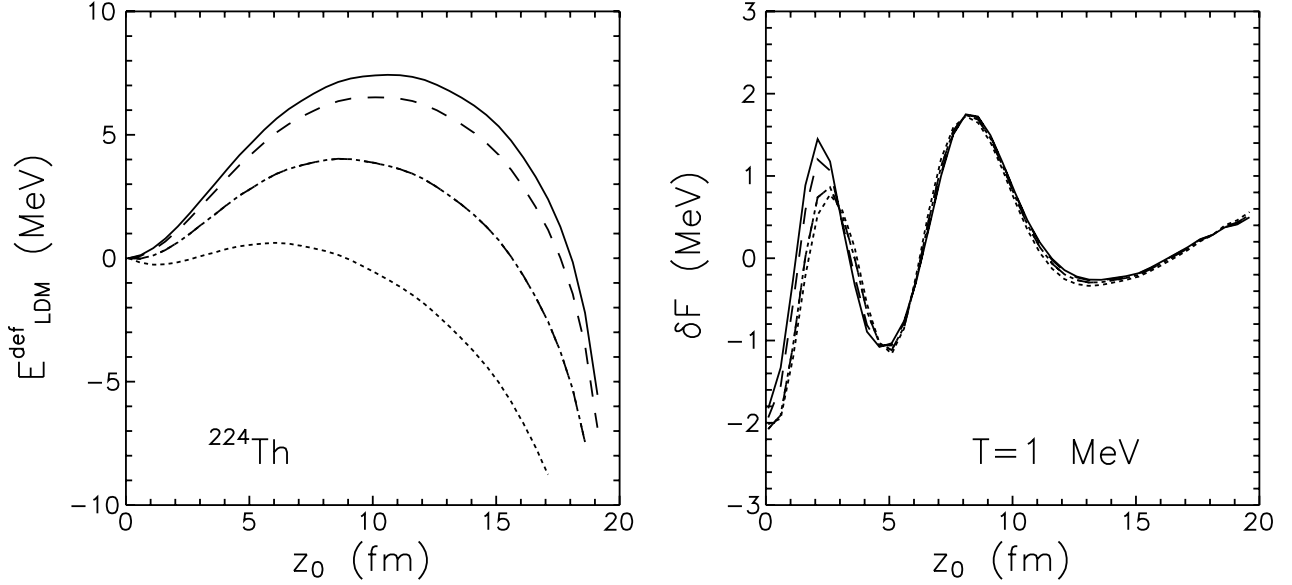


Figure 2: The liquid drop deformation energy (left) and the shell correction $\delta\mathcal{F} = \delta R - T\delta S$ to the free energy (right) for temperature $T = 1 \text{ MeV}$ as function of the deformation parameter z_0 . The solid, dashed, dotted-dashed and dotted lines correspond to the values of angular momentum equal to 0, 20, 40 and $60\hbar$.

4.1 Intrinsic response function

The Fourier transform of the intrinsic response function given by (6) can be expressed as the sum over single-particle states

$$\chi_{\mu\nu}(\omega) = \sum_{jk} \chi_{jk}(\omega) F_{jk}^{\mu} F_{kj}^{\nu} \quad (17)$$

with

$$\chi_{jk}(\omega) = - \int_{-\infty}^{\infty} \frac{d\Omega}{2\pi\hbar} n(\Omega) \left(\varrho_k(\Omega) \mathcal{G}_j(\Omega + \omega + i\epsilon) + \varrho_j(\Omega) \mathcal{G}_k(\Omega - \omega - i\epsilon) \right) \quad (18)$$

Here $n(\Omega)$ is the Fermi function determining the occupation of the (rotation dependent) single-particle levels. The \mathcal{G}_k appearing in (18) is the one-body Green function

$$\mathcal{G}_k(\omega \pm i\epsilon) = \frac{1}{\hbar\omega - \epsilon_k - \Sigma'(\omega, T) \pm i\Gamma(\omega, T)/2} \quad (19)$$

It is parameterized in terms of the real and imaginary part of the self-energy $\Sigma(\omega, T) = \Sigma'(\omega, T) - i\Gamma(\omega, T)/2$. The $\Gamma(\omega, T)$ is assumed to have the form

$$\Gamma(\omega, T) = \frac{1}{\Gamma_0} \frac{(\hbar\omega - \mu)^2 + \pi^2 T^2}{1 + [(\hbar\omega - \mu)^2 + \pi^2 T^2]/c^2} \quad (20)$$

and $\Sigma'(\omega, T)$ is coupled to $\Gamma(\omega, T)$ by the Kramers-Kronig relation. The $\varrho_k(\omega)$ represents the distribution of single-particle strength over more complicated states. It is related to \mathcal{G}_k by

$$\varrho_k(\omega) = i(\mathcal{G}_k(\omega + i\epsilon) - \mathcal{G}_k(\omega - i\epsilon)) \quad (21)$$

For the simplified case when the collisional damping could be neglected the intrinsic response function attains the form

$$\chi_{\mu\nu}(\omega) = \sum_{kl} \frac{n_k - n_l}{\hbar(\omega - \omega_{kl}) + i0} F_{lk}^\mu F_{kl}^\nu \quad (22)$$

with $\hbar\omega_{kl} \equiv \epsilon_k - \epsilon_l$ and $i0$ being infinitely small imaginary constant.

Fig.3 shows few examples of the response function (17)-(21) for the elongation mode. The deformation parameter in this case is the distance $q = q(z_0, \delta)$ between the left and right centers of mass (divided by the diameter of the sphere with the same volume) and \hat{F}_q -operator is, see [9]

$$\hat{F}_q \equiv \frac{\partial \hat{H}}{\partial z_0} \frac{\partial z_0}{\partial q} + \frac{\partial \hat{H}}{\partial \delta} \frac{\partial \delta}{\partial q} = \left(\frac{\partial q}{\partial z_0} + \frac{\partial \delta}{\partial z_0} \frac{\partial q}{\partial \delta} \right)^{-1} \left(\frac{\partial \hat{H}}{\partial z_0} + \frac{\partial \delta}{\partial z_0} \frac{\partial \hat{H}}{\partial \delta} \right) \quad (23)$$

where the derivative $\partial \delta / \partial z_0$ is to be taken along the fission path $\delta = \delta(z_0)$.

Comparing the response function corresponding to different values of ω_{rot} one notices a peak in the low frequency region. This peak is absent in the case $\omega_{rot} = 0$. With growing ω_{rot} the peak gets "stronger" and moves away from $\omega = 0$. Its position is approximately proportional to ω_{rot} . This circumstance hints that this additional peak may be caused by rotation (let us call it here "rotational" peak). Recalling that the friction coefficient γ and mass parameter M (at least in the zero frequency limit) are defined by the derivatives of response function with respect to ω at $\omega = 0$ it is clear that the value of both γ and M can be very sensitive to the "rotational" peak. The numerical results show that the contribution from "rotational" peak to the real part of the response function can lead to a *negative* value of mass parameter. Thus, the "rotational" peak could be of spurious origin and one has to treat this problem very accurately in order to get the reliable results for friction and inertia.

4.2 The conservation of angular momentum

We will consider in this section the case of a single deformation parameter Q for simplicity. The generalization to multi-dimensional case is straightforward.

It is clear that the Routhian $R(Q, \omega_{rot}) = \hat{H}(Q) - \omega_{rot} \hat{J}_x$ violates the rotational symmetry. The operator \mathbf{J} of angular momentum does not commute with Routhian, thus J is not a good quantum number. By varying ω_{rot} one can fix the *average* value of \hat{J}_x , i.e. one chooses ω_{rot} in such a way that

$$\langle \omega_{rot} | \hat{J}_x | \omega_{rot} \rangle = I \quad (24)$$

where by $|\omega_{rot}\rangle$ we denote the Irast state of Routhian (1) calculated for given ω_{rot} .

If in addition to rotation we switch on also the vibrations

$$\hat{H}(Q) - \omega_{rot} \hat{J}_x \implies \hat{H}(Q^0) - \omega_{rot} \hat{J}_x + \hat{F} \delta Q(t) \quad (25)$$

then $|\omega_{rot}\rangle$ becomes time-dependent, $|\omega_{rot}\rangle \implies |\omega_{rot}\rangle_t$ and, in principle, average value of \hat{J}_x is not conserved any more

$$\langle \omega_{rot} | \hat{J}_x | \omega_{rot} \rangle_t = I(t) = I + \delta \langle \hat{J}_x \rangle_t \neq I \quad (26)$$

The variation $\delta \langle \hat{J}_x \rangle_t$ is given within the linear response theory by (9), namely

$$\delta \langle \hat{J}_x \rangle_t = - \int_{-\infty}^{\infty} \tilde{\chi}_{J_x F}(t-s) (Q(s) - Q^0) ds \quad (27)$$

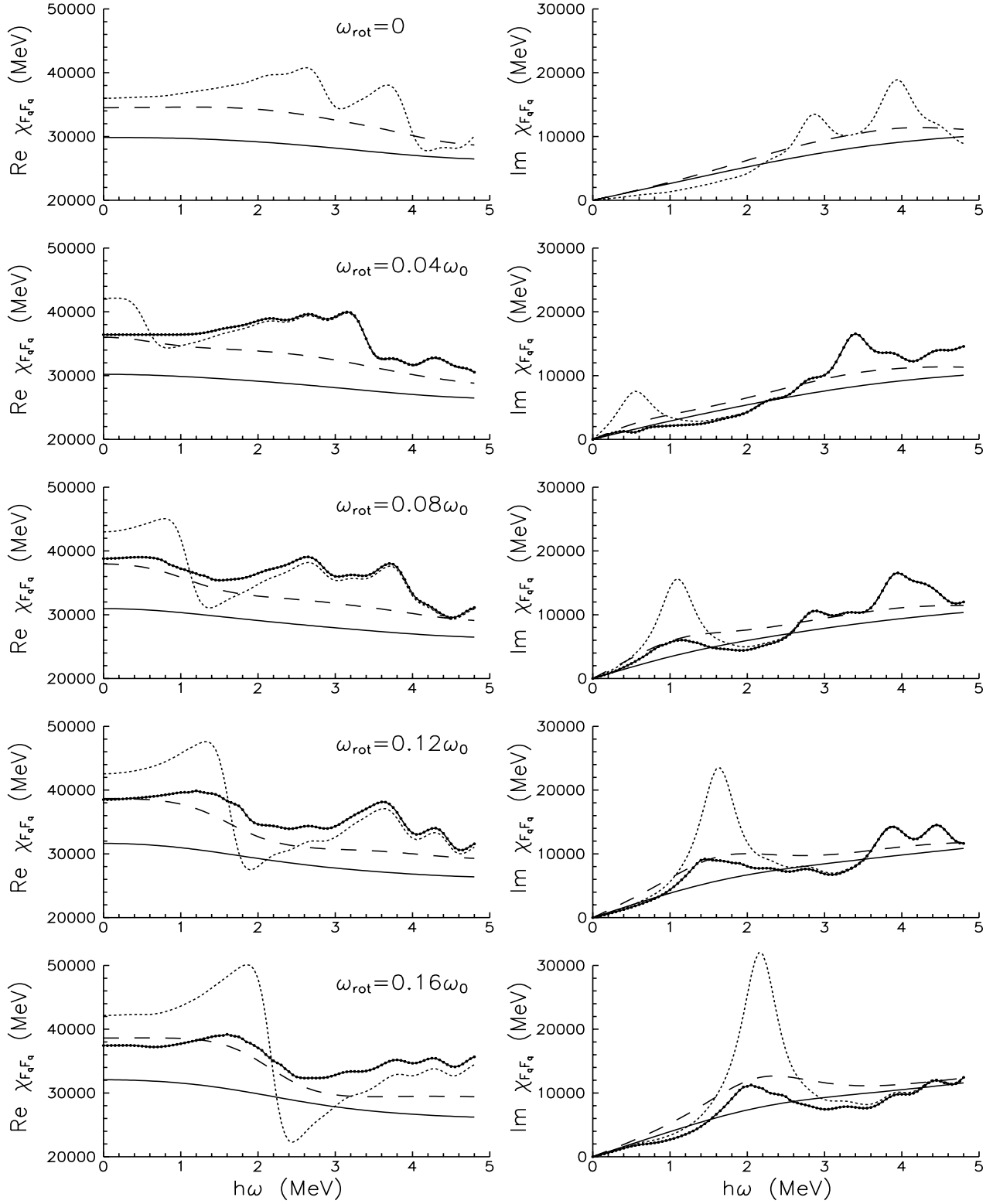


Figure 3: The frequency dependence of the real (left) and imaginary (right) parts of $F_q F_q$ intrinsic response function for several values of the rotational frequency ω_{rot} . The dotted, dashed and solid curves correspond to temperatures $T = 1, 2$ and 3 MeV. The heavy solid curve marks the modified response function $\hat{\chi}_{FF}(\omega)$, see (35), for $T = 1$ MeV. The computations are done for the "ground state" shape of nucleus ^{224}Th very close to the sphere, $z_0 = 0.1$, parameters of spheroidal deformations $\delta_1 = \delta_2 = 0.05$.

The $\delta\langle\hat{J}_x\rangle_t$ given by (27) is not zero. The possible way to make $\langle\omega_{rot}|\hat{J}_x|\omega_{rot}\rangle_t$ time independent is to allow ω_{rot} to depend on time

$$\begin{aligned}\omega_{rot} &\implies \omega_{rot}(t) = \omega_{rot}^0 + \delta\omega_{rot}(t) \\ \hat{H}(Q) - \omega_{rot}\hat{J}_x &\implies \hat{H}(Q^0) - \omega_{rot}^0\hat{J}_x + \hat{F}\delta Q(t) - \hat{J}_x\delta\omega_{rot}(t)\end{aligned}\quad (28)$$

The time-dependent correction $\delta\omega_{rot}(t)$ should be found from the requirement that $\delta\langle\hat{J}_x\rangle_t$ (or its Fourier transform $\delta\langle\hat{J}_x\rangle_\omega$) is equal to zero,

$$\delta\langle\hat{J}_x\rangle_t = \delta\langle\hat{J}_x\rangle_\omega = 0 \quad (29)$$

This problem can be easily solved by means of linear response theory. Considering the time-dependent part $\hat{F}\delta Q(t) - \hat{J}_x\delta\omega_{rot}(t)$ as a small perturbation one can find (mind (9))

$$\begin{aligned}\delta\langle\hat{J}_x\rangle_t &= -\int_{-\infty}^{\infty} \tilde{\chi}_{J_x F}(t-s)(Q(s) - Q^0)ds \\ &\quad - \int_{-\infty}^{\infty} \tilde{\chi}_{J_x J_x}(t-s)(\omega_{rot}(s) - \omega_{rot}^0)ds,\end{aligned}\quad (30)$$

or its Fourier transform

$$\delta\langle\hat{J}_x\rangle_\omega = -\chi_{J_x F}(\omega)\delta Q(\omega) - \chi_{J_x J_x}(\omega)\delta\omega_{rot}(\omega) \quad (31)$$

The analogous expression can be also written for the variation $\delta\langle F\rangle_\omega$

$$\delta\langle F\rangle_\omega = -\chi_{FF}(\omega)\delta Q(\omega) - \chi_{F J_x}(\omega)\delta\omega_{rot}(\omega) \quad (32)$$

Recalling (29) one can find $\delta\omega_{rot}(\omega)$ from (31) as

$$\delta\omega_{rot}(\omega) = -\chi_{J_x F}(\omega)\delta Q(\omega)/\chi_{J_x J_x}(\omega) \quad (33)$$

In principle, $\delta\omega_{rot}(t)$ could be found by Fourier transform of (33). But this is not necessary for practical purpose. One can insert $\delta\omega_{rot}(\omega)$ in (32) to define the modified response function $\hat{\chi}_{FF}(\omega)$,

$$\delta\langle F\rangle_\omega = -\hat{\chi}_{FF}(\omega)\delta Q(\omega) \quad (34)$$

with

$$\hat{\chi}_{FF}(\omega) = \chi_{FF}(\omega) - \frac{\chi_{F J_x}(\omega)\chi_{J_x F}(\omega)}{\chi_{J_x J_x}(\omega)} \quad (35)$$

The modified response function (35) is shown by heavy solid line in Fig.3. It is seen that "rotational peak" has disappeared. Consequently, the transport coefficients computed with modified response function (35) will not contain the contributions from the "rotational peak" and would differ considerably from these derived with $\chi_{FF}(\omega)$.

The above method was successfully used in [19] to remove the contributions to the response function caused by the violation of the particle number conservation by pairing. It was demonstrated there that fixing of the average value of the particle number with the time-dependent density matrix leads to the same secular equation for the vibrational mode as obtained within RPA. This method is rather general and can be used to fix any physical quantity. For example, using transformation analogous to (35) one can remove the center of mass motion in case of isoscalar dipole vibrations.

It is of certain interest to compare the secular equation which results from the above approach with one obtained earlier, for example within the so called "cranked RPA" [20, 21].

It is argued in [20, 21] that for the description of non-rotational excitations in the rotating nuclei one should substitute the Routhian (1) by some supplementary rotational invariant Hamiltonian \widetilde{H}

$$\hat{R} = \hat{H} - \omega_{rot} \hat{J}_x \implies \widetilde{H} = \hat{H} - h(\hat{\mathbf{J}}^2) \quad (36)$$

The many-body Hamiltonian \hat{H} was approximated in [20, 21] by the mean-field part \hat{H}_0 plus quadrupole-quadrupole interaction

$$\hat{H} = \hat{H}_0 - \frac{\kappa}{2} \sum_{m=-2}^{m=2} (-1)^m \hat{Q}_{2m} \hat{Q}_{2-m} \quad (37)$$

and for $h(\hat{\mathbf{J}}^2)$ the expansion was used

$$h(\hat{\mathbf{J}}^2) = \langle h \rangle + \omega_{rot}(\hat{J}_x - \langle \hat{J}_x \rangle) + \mu_x(\hat{J}_x - \langle \hat{J}_x \rangle)^2 + \mu(\hat{J}_y^2 + \hat{J}_z^2) + \dots \quad (38)$$

with

$$\omega_{rot} = 2\mu \langle \hat{J}_x \rangle, \quad \mu_x = \frac{1}{2} d^2 \langle \hat{H} \rangle / d \langle \hat{J}_x \rangle^2 \quad (39)$$

The terms omitted in (38) are of the higher order in $\hat{J}_x - \langle \hat{J}_x \rangle$, \hat{J}_y or \hat{J}_z . Solving of the Hamiltonian \widetilde{H} within RPA leads to the secular equation

$$\omega^2 \mathcal{F}^+(\omega) = 0 \quad (40)$$

with

$$\mathcal{F}^+(\omega) = \begin{vmatrix} S_{xx} & S_{x0} & S_{x2} \\ S_{0x} & S_{00} - \kappa/2 & S_{02} \\ S_{2x} & S_{20} & S_{22} - \kappa/2 \end{vmatrix} \quad (41)$$

where

$$S_{\mu\nu} = \sum_{k>l} \left\{ \frac{(n_k - n_l) \omega_{kl}}{\omega^2 - \omega_{kl}^2} q_{kl}^{(\mu)} q_{kl}^{(\nu)} + \frac{(n_{\bar{k}} - n_{\bar{l}}) \omega_{\bar{k}\bar{l}}}{\omega^2 - \omega_{\bar{k}\bar{l}}^2} q_{kl}^{(\mu)} q_{kl}^{(\nu)} \right\} \quad (42)$$

The spurious zero energy state is separated by (40) and all other excitations are given by $\mathcal{F}^+(\omega) = 0$.

To get the equation for the collective frequencies in our approach let us recall that the equation (29) together with the self-consistency condition

$$\delta \langle \hat{F} \rangle_\omega = k \delta Q(\omega) \quad (43)$$

leads to the system of equations (mind (32),(31))

$$\begin{aligned} (k + \chi_{FF}(\omega)) \delta Q(\omega) + \chi_{FJ_x}(\omega) \delta \omega_{rot}(\omega) &= 0 \\ \chi_{J_x F}(\omega) \delta Q(\omega) + \chi_{J_x J_x}(\omega) \delta \omega_{rot}(\omega) &= 0 \end{aligned} \quad (44)$$

The eigenfrequencies for the system (44) are found from the equation

$$\text{Det}(\omega) \equiv (k + \chi_{FF}(\omega))\chi_{J_x J_x}(\omega) - \chi_{FJ_x}(\omega)\chi_{J_x F}(\omega) = 0 \quad (45)$$

If one would neglect the effects of collisional damping then one could use expressions (22) for the response functions. In this case $\text{Det}(\omega)$ coincides exactly with the nontrivial part $\mathcal{F}^+(\omega)$ of the secular equation (40) obtained in [20] (one should also put $S_{2\nu} = S_{\mu 2} = 0$ since only axially symmetric shapes are considered in present work). So Eq.(45) does not contain the spurious contributions caused by the violation of rotational symmetry.

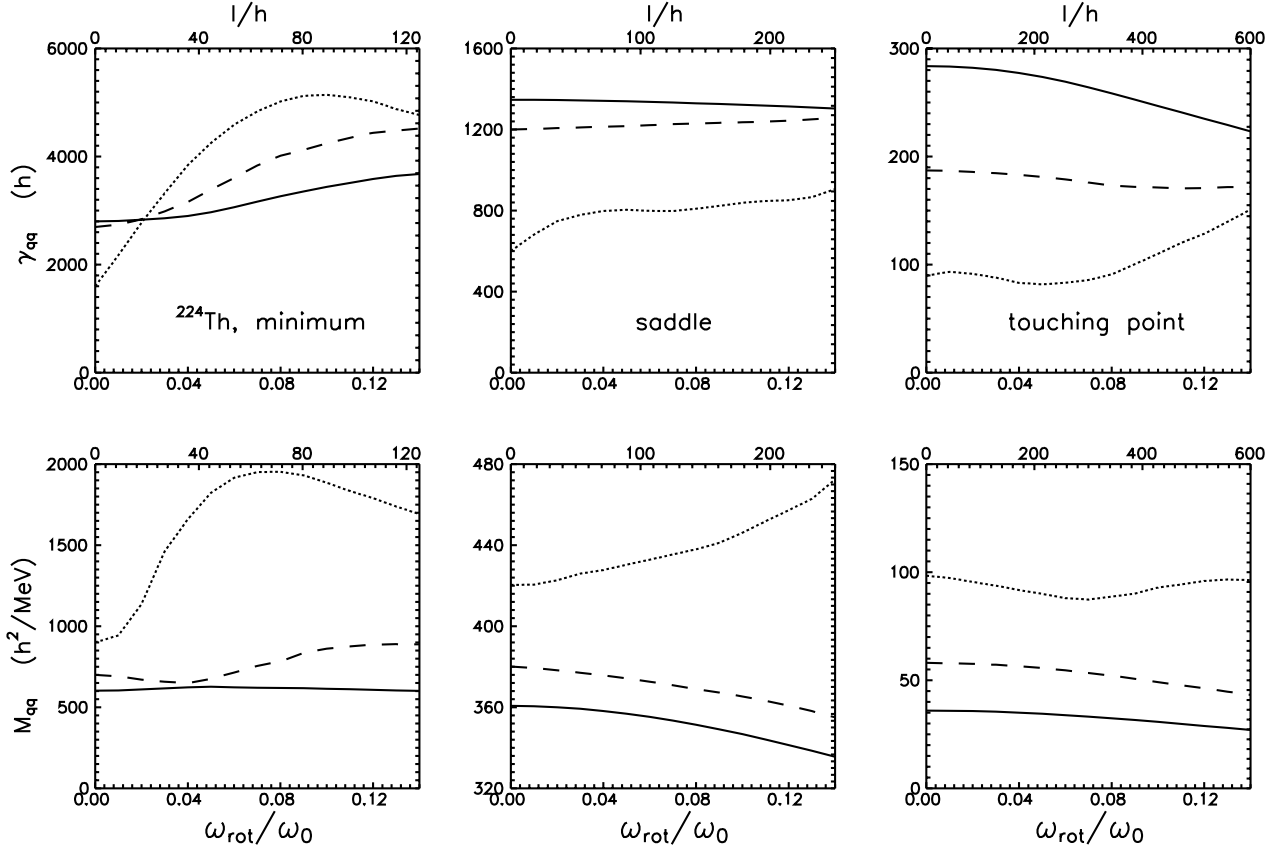


Figure 4: The friction and inertia (5) for the elongation mode as functions of the average value of angular momentum I (upper x -axes) or rotational frequency ω_{rot} (lower x -axes). Dotted, dashed and solid curves correspond to temperatures $T = 1, 2$ and 3 MeV . The computations are done for several deformations of nucleus ^{224}Th which correspond to the minimum of potential energy, saddle and two touching spheres.

5 Transport coefficients

In Fig.4 we show the friction γ_{qq} and mass M_{qq} coefficients defined by the fit (5) of the collective response function for three particular deformations which are of a special interest: at the ground state deformation of ^{224}Th , at the saddle and for at touching point. The last configuration is of the interest for description of initial stage of fusion reactions. The deformation parameter $q \equiv R_{12}/2R_0$ is here the distance R_{12} (divided by the diameter of

nucleus) between the centers of mass of left and right parts of nucleus. The advantage of such a choice is explained in [9, 10]. As it is seen from Fig.4 the dependence of γ_{qq} on ω_{rot} is much weaker as that found in [4]. Evidently, this is because we have removed the spurious "rotational" peak from the response function. Without modification (35) we would obtain the friction coefficient which look very much like that of [4].

One can see from Fig.4 that for the ground state deformation and temperatures $T = 1\text{MeV}$ the friction and mass coefficients depend somewhat on the rotation. Both γ_{qq} and M_{qq} increase with ω_{rot} in the interval $0 \leq \omega_{rot} \leq 0.08\omega_0$. For higher T and ω_{rot} both γ_{qq} and M_{qq} are rather stable with respect to variation of T and ω_{rot} . For more deformed shapes the friction and mass coefficients are not very sensitive to the rotation for all temperatures.

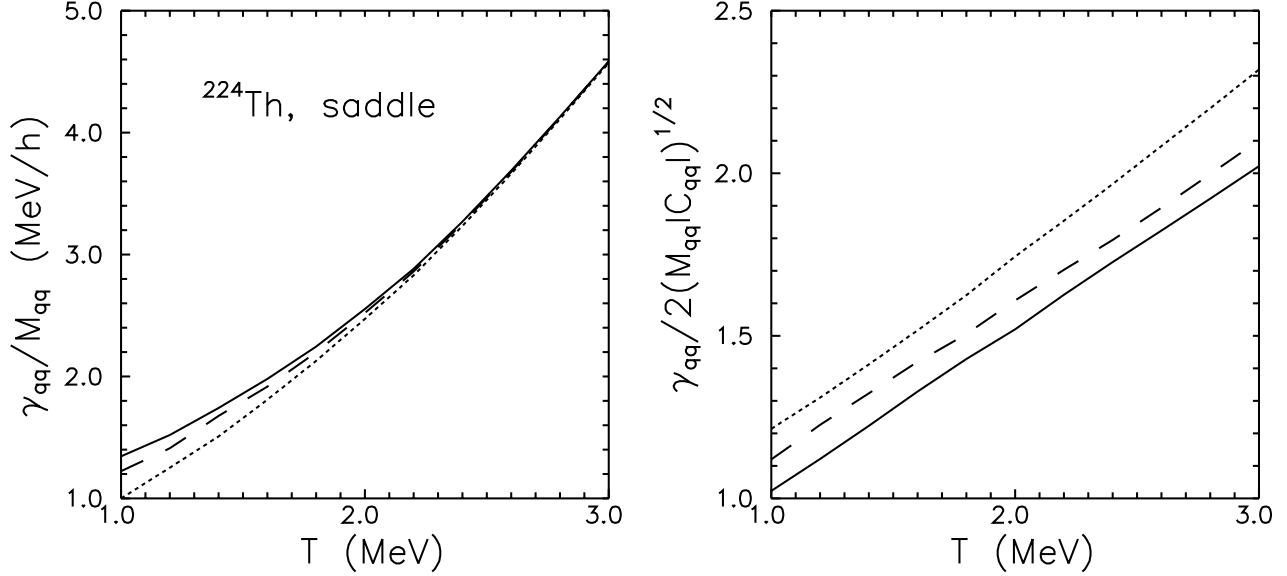


Figure 5: The reduced friction coefficient $\beta_{qq} = \gamma_{qq}/M_{qq}$ (left) and the damping factor $\eta_{qq} = \gamma_{qq}/2\sqrt{|C_{qq}|M_{qq}}$ (right) versus temperature. The dotted, dashed and solid curves correspond to the values of angular momentum equal to 0, 40 and 60 \hbar .

Finally, Fig.5 shows the reduced friction coefficient $\beta_{qq} = \gamma_{qq}/M_{qq}$ and the damping factor $\eta_{qq} = \gamma_{qq}/2\sqrt{|C_{qq}|M_{qq}}$ at the saddle of ^{224}Th as the function of temperature. The damping factor reveals whether collective motion is underdamped ($\eta < 1$) or overdamped ($\eta > 1$). As Fig.5 shows, the collective motion changes from underdamped to overdamped at $T \approx 1\text{MeV}$. Both β_{qq} and η_{qq} shown in Fig.5 increase with the temperature. This behaviour is in a qualitative agreement with the one found in [8]. The increase of η_{qq} with the temperature is impossible to explain neither with the wall friction nor with the hydrodynamical viscosity. The increase of η_{qq} with temperature obtained here is not as rapid as that found in [8]. The account of rotation does not diminish this discrepancy since both β_{qq} and η_{qq} shown in Fig.5 do not depend much on the rotation. The dependence of η_{qq} on I seen from Fig.5 is mainly due to some dependence of liquid drop stiffness on rotation. We should also note that in this work we did not normalize the high temperature limit of the mass parameter to the irrotational flow value. That is why the numerical results for the mass parameter and, consequently, reduced friction β_{qq} and damping factor η_{qq} differ somewhat from these obtained in [9].

6 Summary and outlook

We have examined the influence of rotation on the transport coefficients of the collective motion. Rather unexpectedly we have found out that friction γ and mass M parameters for rotating nuclei are rather sensitive to such fine effects as the violation of rotational symmetry by Coriolis term $-\omega_{rot}\hat{J}_x$. For the ground state deformation the spurious contributions to collective friction and mass are (at least) as large as those of physical importance. This circumstance was not clear (to the best of our knowledge) up to now.

In order to remove the spurious contributions we had to modify the model of "stationary rotation" and to introduce the time-dependent rotational frequency. In the result we obtained the friction and the mass which demonstrate rather reasonable dependence on the rotational frequency ω_{rot} . For excitations above $T = 2MeV$ when the microscopic shell effects disappear both γ and M are rather insensitive to the rotation, i.e. behave like macroscopic quantities. For $1MeV \leq T \leq 2MeV$ we found some increase of γ and M with growing ω_{rot} . Such effect might be caused by the change of shell structure due to the re-arrangement of single-particle states by rotation.

Even stronger dependence of γ and M on ω_{rot} should be expected for $T \leq 1MeV$ when both shell end pairing effects are present. As it was shown in [19, 22] the pairing effects change considerably the collective transport at low excitations. The destruction of pairing by the rotation can have considerable effect on the transport coefficients and is worth to be examined. Such details could be very important for the accurate description, for example, of the final stage of the fusion reaction and formation of superheavy elements which takes place at low excitation energy.

The extension of the method developed in the present work to the simultaneous treatment of both pairing and rotation will be the subject of future work.

Acknowledgements. The authors are grateful to H.Hofmann who has drawn their attention to the problem of the dependence of transport coefficients on the angular rotation as well as to I.N.Mikhailov, V.V.Pashkevich and K.Pomorski for fruitful discussions. One of us (F.A.I) would like to thank the Cyclotron Center, RIKEN for the hospitality extended to him during his stay in Japan.

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A The matrix elements of \hat{J}_x operator

The single-particle basis wave functions are defined in the two-center shell model [5, 6] as

$$|n_z n_\rho m s_z\rangle = \varphi_{n_z}(z) \chi_{n_\rho}^{|m|}(\rho) \eta_m(\varphi) \chi_{1/2}(s_z) \quad (46)$$

where

$$\eta_m(\varphi) = \frac{1}{\sqrt{2\pi}} e^{im\varphi}, \quad (47)$$

with m being an arbitrary integer number and

$$\chi_{n_\rho}^{|m|}(\rho) = (-1)^{\frac{m+|m|}{2}} \sqrt{\frac{2(n_\rho)!}{(n_\rho + |m|)!}} k_\rho^{\frac{|m|+1}{2}} e^{-\frac{1}{2}k_\rho \rho^2} \rho^{|m|} L_{n_\rho}^{|m|}(k_\rho \rho^2) \quad (48)$$

Here $k_\rho = m_0 \omega_\rho / \hbar$, n_ρ is a not-negative integer and $L_{n_\rho}^{|m|}(\xi)$ is a Laguerre polynomial. The z -components of wave function

$$\varphi_{n_z}(z) = \begin{cases} N_{n_{z1}}^{-1} U(-n_{z1} - \frac{1}{2}, -\sqrt{2k_{z1}}(z - z_1)) & \text{for } z < 0 \\ N_{n_{z2}}^{-1} U(-n_{z2} - \frac{1}{2}, \sqrt{2k_{z2}}(z - z_2)) & \text{for } z > 0 \end{cases} \quad (49)$$

are expressed in terms of parabolic cylinder functions $U(a, x)$,

$$U(a, x) = \frac{\sqrt{\pi}}{2^{a/2+1/4}} \frac{{}_1F_1(a/2 + 1/4, 1/2, x^2/2)}{\Gamma(a/2 + 3/4)} e^{-x^2/4} - \frac{\sqrt{2\pi}x}{2^{a/2+1/4}} \frac{{}_1F_1(a/2 + 3/4, 3/2, x^2/2)}{\Gamma(a/2 + 1/4)} e^{-x^2/4} \quad (50)$$

The constants $N_{n_{z1}}$ and $N_{n_{z2}}$ are defined by the normalization and the continuity of $\varphi_{n_z}(z)$ and its first derivative at $z = 0$, see [5].

The operator of single-particle angular momentum $\hat{\mathbf{j}}$ is

$$\hat{\mathbf{j}} = \hat{\mathbf{l}} + \hat{\mathbf{s}} \quad (51)$$

where $\hat{\mathbf{l}}$ is the orbital momentum $\hat{\mathbf{l}} = -i [\mathbf{r} \nabla]$ and $\hat{\mathbf{s}}$ is the spin $\hat{\mathbf{s}} = \frac{1}{2} \boldsymbol{\sigma}$, with $\boldsymbol{\sigma}$ being the Pauli matrices. From (51) it follows immediately that

$$\langle n_z n_\rho m \ s_z | \hat{j}_x | n'_z n'_\rho m' \ s'_z \rangle = \langle n_z n_\rho m \ | \hat{l}_x | n'_z n'_\rho m' \rangle \delta_{s_z, s'_z} + \langle s_z | \hat{s}_x | s'_z \rangle \delta_{n_z, n'_z} \delta_{n_\rho, n'_\rho} \delta_{m, m'} \quad (52)$$

For the spin part of (52) it is easy to find

$$\langle s_z | \hat{s}_x | s'_z \rangle = \frac{1}{2} (\delta_{s'_z, s_z+1} + \delta_{s'_z, s_z-1}) \quad (53)$$

The x -component of the orbital momentum \hat{l}_x is given in the cylindrical co-ordinate system $\{r, \theta, \varphi\}$ by

$$\hat{l}_x = i \left\{ \left(z \frac{\partial}{\partial \rho} - \rho \frac{\partial}{\partial z} \right) \sin \varphi + \frac{z \cos \varphi}{\rho} \frac{\partial}{\partial \varphi} \right\} \quad (54)$$

Note that $\hat{l}_x^* = -\hat{l}_x$.

If the nucleus is not left-right symmetric then the rotation axes does not go through $z = 0$ but through the center of mass z_{cm} . Consequently eq.(54) should be modified to

$$\hat{l}_x = i \left\{ \left((z - z_{cm}) \frac{\partial}{\partial \rho} - \rho \frac{\partial}{\partial z} \right) \sin \varphi + \frac{(z - z_{cm})}{\rho} \cos \varphi \frac{\partial}{\partial \varphi} \right\} \quad (55)$$

Since the single particle wave functions are separable in $\{\rho, z, \varphi\}$ the matrix elements $\langle n_z n_\rho m \ | \ l_x \ | \ n'_z n'_\rho m' \rangle$ are then the product of one-dimensional matrix elements

$$\begin{aligned} \langle n_z n_\rho m \ | \ l_x \ | \ n'_z n'_\rho m' \rangle = & \langle n_\rho m \ | \ \rho \ | \ n'_\rho m' \rangle \langle m \ | \ -i \sin \varphi \ | \ m' \rangle \langle n_z \ | \ \frac{\partial}{\partial z} \ | \ n'_z \rangle + \langle n_z \ | \ z - z_{cm} \ | \ n'_z \rangle \times \\ & \left[\langle n_\rho m \ | \ \frac{\partial}{\partial \rho} \ | \ n'_\rho m' \rangle \langle m \ | \ i \sin \varphi \ | \ m' \rangle + \langle n_\rho m \ | \ \frac{1}{\rho} \ | \ n'_\rho m' \rangle \langle m \ | \ i \cos \varphi \frac{\partial}{\partial \varphi} \ | \ m' \rangle \right] \end{aligned} \quad (56)$$

The matrix elements $\langle n_z \ | \ z - z_{cm} \ | \ n'_z \rangle$ and $\langle n_z \ | \ \partial/\partial z \ | \ n'_z \rangle$ are the same as computed in the two center shell model code [5, 6]. For $\langle m \ | \ -i \sin \varphi \ | \ m' \rangle$ and $\langle m \ | \ i \cos \varphi \partial/\partial \varphi \ | \ m' \rangle$ it is easy to find

$$\begin{aligned} \langle m \ | \ -i \sin \varphi \ | \ m' \rangle &= \frac{1}{2} (\delta_{m', m+1} - \delta_{m', m-1}), \\ \langle m \ | \ i \cos \varphi \frac{\partial}{\partial \varphi} \ | \ m' \rangle &= \frac{-m'}{2} (\delta_{m', m+1} + \delta_{m', m-1}) \end{aligned} \quad (57)$$

What is left to calculate are "ρ"-matrix elements. These can be calculated using the recurrence relation between Laguerre polynomials and their derivatives. After somewhat lengthy derivation one can find

$$\begin{aligned} & \langle n_z n_\rho m | \hat{l}_x | n'_z n'_\rho m' \rangle = \\ & \frac{1}{2} \left[\sqrt{n_\rho + m} \delta_{m', m-1} + \sqrt{n_\rho} \delta_{m', m+1} \right] \left[\sqrt{k_\rho} \langle n_z | z - z_{cm} | n'_z \rangle + \frac{1}{\sqrt{k_\rho}} \langle n_z | \frac{\partial}{\partial z} | n'_z \rangle \right] \delta_{N'_\rho, N_\rho-1} + \\ & \frac{1}{2} \left[\sqrt{n'_\rho + m'} \delta_{m', m+1} + \sqrt{n'_\rho} \delta_{m', m-1} \right] \left[\sqrt{k_\rho} \langle n_z | z - z_{cm} | n'_z \rangle - \frac{1}{\sqrt{k_\rho}} \langle n_z | \frac{\partial}{\partial z} | n'_z \rangle \right] \delta_{N'_\rho, N_\rho+1} \end{aligned} \quad (58)$$

Here we have introduced the quantum number $N_\rho = 2n_\rho + |m|$. The expression (58) is valid for m and m' being it both non-negative. The matrix elements for non-positive m and m' can be related to (58) using (47) and symmetry properties of \hat{l}_x

$$\langle n_z n_\rho - m | l_x | n'_z n'_\rho - m' \rangle = \langle n_z n_\rho m | l_x | n'_z n'_\rho m' \rangle, \quad (59)$$

The operator \hat{j}_x couples the states with $\Delta j_z = \pm 1$. In this way the states with positive and negative j_z are coupled to each other. One can reduce the dimension of matrix to be diagonalized by factor two using so called Goodman transformation [23]. It was suggested in [23] to introduce the basis states of the type

$$| K \rangle = \frac{1}{\sqrt{2}}(| k \rangle + | \bar{k} \rangle), \quad | \bar{K} \rangle = \frac{1}{\sqrt{2}}(| \bar{k} \rangle - | k \rangle) \quad (60)$$

where $| k \rangle = | n_z n_\rho m s_z \rangle$ for such m and s_z that $m + s_z - 1/2 \equiv j_z - 1/2$ is even. The single particle states $| n_z n_\rho m s_z \rangle$ with $j_z - 1/2$ - odd up to a sign factor coincide with $| \bar{k} \rangle$. It was shown in [23] that the matrix elements $\langle K | \hat{j}_x | \bar{K}' \rangle$ are zero and the matrix of \hat{j}_x on the states $| K \rangle$ is of quasi-diagonal form. The nonzero matrix elements are

$$\langle K | \hat{j}_x | K' \rangle = -\langle \bar{K} | \hat{j}_x | \bar{K}' \rangle = \langle k | \hat{j}_x | \bar{k}' \rangle \quad (61)$$

So the quantities of interest are matrix elements $\langle k | \hat{j}_x | \bar{k}' \rangle \equiv \langle k | \hat{j}_x T | k' \rangle$. It turns out possible to express the matrix elements $\langle K | \hat{j}_x | K' \rangle$ in terms of $\langle k | \hat{j}_x | k' \rangle$,

$$\langle K | \hat{j}_x | K' \rangle = \langle n_z n_\rho m s_z | \hat{j}_x | n'_z n'_\rho m' s'_z \rangle \quad (62)$$

For $\langle n_z n_\rho m s_z | \hat{j}_x | n'_z n'_\rho m' s'_z \rangle$ the expressions (52),(53),(58) are to be used.

The particular case of $m + s_z = m' + s'_z = 1/2$ should be considered separately. In this case one has to combine (58) with (59) to obtain

$$\begin{aligned} & \langle n_z, n_\rho, 1/2 - s_z, s_z | j_x T | n'_z, n'_\rho, 1/2 - s'_z, s'_z \rangle = \frac{1}{2} \delta_{n_z, n'_z} \delta_{n_\rho, n'_\rho} \delta_{s'_z, 1-s_z} \\ & + \frac{1}{2} \delta_{s'_z, -s_z} \delta_{N'_\rho, N_\rho-1} \sqrt{n_\rho + 1/2 - s_z} \left[\sqrt{k_\rho} \langle n_z | z - z_{cm} | n'_z \rangle + \frac{1}{\sqrt{k_\rho}} \langle n_z | \frac{\partial}{\partial z} | n'_z \rangle \right] \\ & + \frac{1}{2} \delta_{s'_z, -s_z} \delta_{N'_\rho, N_\rho+1} \sqrt{n'_\rho + 1/2 - s'_z} \left[\sqrt{k_\rho} \langle n_z | z - z_{cm} | n'_z \rangle - \frac{1}{\sqrt{k_\rho}} \langle n_z | \frac{\partial}{\partial z} | n'_z \rangle \right] \end{aligned} \quad (63)$$